

Singularity and Symmetry Analyses for Tuberculosis Epidemics

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Abstract. We analyse the model of Tuberculosis due to Blower (*Nature Medecine* **1**(8) 815-821) from the point of view of symmetry and singularity analysis. From the study we provide a demonstration of the integrability of the model to present an explicit solution.

Key words: Singularity; Symmetry; TB epidemics

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1 Introduction

Tuberculosis (TB) is an airborne-transmitted disease and in human beings is caused by *Mycobacterium tuberculosis* bacteria (Mtb). Mtb droplets are released into the air by an infectious individual coughing and/or sneezing. Tubercle bacillus carried by such droplets live in the air for a short period of time [19] (about 2 hours) and therefore it is believed that occasional contact with an infectious person rarely leads to infection. TB is described as a slow disease because of its long and variable latency period and its short and relatively narrow infectious period [19]. The initially exposed individuals (infected individuals) have a higher risk of developing active TB [3]. These individuals still face the possibility of progressing to infectious TB, but the rate of progression slows. In other words the likelihood of becoming an active infectious case decreases with the age of the infection. With this in mind several researchers constructed a series of dynamical models for TB progression and transmission in scenarios that took these factors into consideration [3, 4, 5].

In this paper we analyse the model formulated by Blower *et al* [3] from the point of view of Singularity and Lie analysis. Blower *et al* divided the population of interest into three epidemiological classes: susceptible, latent and infectious. The infection rate given by βSI (using the law of mass action) is divided. A portion, $p\beta SI$, gives rise to immediately active cases (fast progression) while the rest, $(1 - p)\beta SI$, gives rise to latent-TB cases with a low risk of progressing to active TB (slow progression) [3]. The rate of progression from latent TB to active TB is assumed to be proportional to the number of latent-TB cases, that is, it is given by kE , where k ranges from 0.00256 to 0.00527 (slow progression) [3]. The total incidence rate is $p\beta SI + kE$. In our analysis we focused on the situation in which the rate of recruitment is equal to the birth rate and the total population size is constant [3]. Thus the model system is

given by [3]

$$\begin{aligned}\frac{dS}{dt} &= \mu - \beta S(t)I(t) - \mu S(t), \\ \frac{dE}{dt} &= (1-p)\beta S(t)I(t) - kE(t) - \mu E(t) \\ \frac{dI}{dt} &= p\beta S(t)I(t) + kE(t) - \mu I(t).\end{aligned}\tag{1}$$

This paper is organised as followed. In Section 2 we subject the model system (1) to the Painlevé analysis and we present a model as a raw dynamical system which a single second order differential equation. We perform a Lie symmetry analysis in Section 3 and obtained the explicit solutions in the case where the infection rate is the sum of the death rate and the rate of progression from latent TB to active TB. In Section 4 we study in detail and plot the solutions by using the link between parameters that the Lie group analysis has given.

2 Singularity Analysis

There are four standard approaches to the analysis of nonlinear ordinary or partial differential equations. The approaches comprise numerical computation, dynamical systems analysis, singularity analysis and symmetry analysis, all of which possess extensive literatures. Singularity analysis was initiated by Kowalevski [14] in her determination of the third integrable case of the Euler equations for the top and was in large measure developed by the French School developed by Paul Painlevé about the period of La Belle Epoque [8, 9, 6]. There have been significant contributions since then. For a recent and an erudite contribution to the state of the art see the book edited by Conte [7]. For less technical works devoted to the methodology the interested reader is referred to the text of Tabor [20] and the report of Ramani *et al.* [18]. The essence of the singularity analysis of a differential equation is the determination of the existence of isolated movable polelike singularities about which one can develop a Laurent expansion containing arbitrary constants equal in number to the order of the system [17]. The location of the singularity is determined by the initial conditions of the system. An equation of moderately, or more, complicated structure can possess more than one polelike singularity [17].

The application of the analysis is usually quite algorithmic [17]. Indeed, it is standard practice to apply ARS algorithm [1], although there are instances, of particular relevance to the analysis of systems of first-order ordinary linear differential equations typically encountered in the mathematical modelling of epidemics [17], in which the subtler approach advocated by Hua *et al.* [13] is to be preferred [17]. The singularity analysis is a powerful tool for construction of symmetries, explicit solutions and Lie-Bäcklund transformation. It is also helps to find Lax pairs and recursion operators and plays an important role in the study of a chaotic behaviour of nonlinear differential equations [16].

2.1 Singularity analysis of the three dimensional system

We begin the singularity analysis of (1) in the usual way by substituting $S = a_0\tau^{p_1}$, $E = b_0\tau^{p_2}$ and $I = c_0\tau^{p_3}$ to determine the leading-order behaviour. We find that

$$p_1 = p_2 = p_3 = -1 \text{ and } \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{p\beta} \\ \frac{1-p}{p\beta} \\ \frac{1}{\beta} \end{pmatrix}. \quad (2)$$

The resonances are given by

$$r = -1 \text{ and } r = 1(2). \quad (3)$$

We check for consistency at the resonance by substituting

$$S = a_i\tau^{i-1}, \quad E = b_i\tau^{i-1} \text{ and } I = c_i\tau^{i-1} \quad (4)$$

into the full system, (1), with the leading-order term as given in (2). At the resonance +1 we obtain the system

$$\begin{pmatrix} 1 & 0 & -\frac{1}{p} \\ 1-p & 0 & -\frac{1-p}{p} \\ p & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} \frac{\mu}{p\beta} \\ \frac{(1-p)(k+\mu)}{p\beta} \\ \frac{\mu}{\beta} - \frac{k(1-p)}{p\beta} \end{pmatrix} \quad (5)$$

The three equations in (5) are identical if and only if $k = 0$. The system (1) is consistent if the transition rate from the exposed class to the infectious class is zero. Therefore (1) passes the Painlevé test and is integrable in the sense of Poincaré for the constraint $k = 0$.

2.2 Singularity analysis of the two dimensional system

By assuming that the total population size is constant, we have

$$N(t) = S(t) + E(t) + I(t) = 1 \equiv \text{Constant}. \quad (6)$$

We derive $I(t)$ from (6), i.e.

$$I(t) = 1 - S(t) - E(t), \quad (7)$$

the three-dimensional system (1) is reduced to the two-dimensional system

$$\dot{S} = \mu + \beta S^2 + \beta SE - (\beta + \mu)S \quad (8)$$

$$\dot{E} = (1-p)\beta S - (1-p)\beta S^2 - (1-p)\beta SE - (k + \mu)E. \quad (9)$$

The exponents for the usual leading-order behaviour substitution, $S = \alpha_1\tau^{q_1}$ and $E = \alpha_2\tau^{q_2}$ are

$$\begin{array}{lcl} q_1 - 1 & : & q_1 \quad 2q_1 \quad q_1 + q_2 \\ q_2 - 1 & : & q_1 \quad 2q_1 \quad q_1 + q_2 \quad q_2 \end{array}$$

The case $q_1 = q_2 = -1$ contains the left-hand side and the second and the third terms of the right-hand side of (8) and the left-hand side and the second and the third terms of the right-hand side of (9) as dominant terms.

The coefficients of the leading-order terms are

$$\alpha_1 = \frac{p-2}{\beta} \quad \alpha_2 = -\frac{p-1}{p\beta} \quad (10)$$

In the case that the leading-order behaviour does not provide the correct number of arbitrary constants the generic situation illustrated by the results given in (10) it is necessary to determine whether there exists a term, or terms, at which the requisite number of arbitrary constants can enter. The powers at which these arbitrary constants enter are almost called resonances on occasion they are also known as Kowalevski exponents (after the pioneering woman in this area). To determine the resonances we substitute

$$S = \alpha_1 + m\tau^{r-1} \quad E = \alpha_2 + n\tau^{r-1} \quad (11)$$

into (8) and (9) and collect the linear terms in m and n we obtain

$$\begin{pmatrix} r-p+2 & -p+2 \\ -(p-1)(p-3) & r-p^2+3p-3 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = 0 \quad (12)$$

The requirement that the system be consistent leads to the equation

$$r^2 - r(p-1)^2 - (2-p)p = 0. \quad (13)$$

For $p = 1$ the solution of (13) is $r = \pm 1$.

The second approach to the determination of the resonances and the question of consistency mentioned in [13] is particularly suited to a system containing a selection of parameters. Typically such systems are only integrable subject to some constraint(s) on the parameters. After the nature of the polelike singularity is identified, we substitute

$$S = \sum_{j=0}^{\infty} a_j \tau^{j-1}, \quad E = \sum_{j=0}^{\infty} b_j \tau^{j-1} \quad (14)$$

to obtain

$$\begin{aligned} \sum_{i=0}^{\infty} \{ (i-1)a_i \tau^{i-2} + (\beta + \mu)a_i \tau^{i-1} - \sum_{j=0}^{\infty} [\beta(a_i a_j + a_i b_j)] \tau^{i+j-2} \} - \mu &= 0 \\ \sum_{i=0}^{\infty} \{ (i-1)b_i \tau^{i-2} - [(1-p)\beta a_i - (k + \mu)b_i] \tau^{i-1} + \sum_{j=0}^{\infty} [(1-p)\beta(a_i a_j + a_i b_j)] \tau^{i+j-2} \} &= 0 \end{aligned}$$

We illustrate the workings of the algorithm [10] with the first few powers.

τ^{-2} :

$$-a_0 - \beta(a_0^2 + a_0 b_0) = 0 \quad (15)$$

$$-b_0 + (1-p)\beta(a_0^2 + a_0 b_0) = 0 \quad (16)$$

from (15) and (16) we have

$$a_0 = \frac{p-2}{\beta}, \quad b_0 = -\frac{p-1}{p\beta} : \text{not a resonance, then we continue}$$

and

$$b_0 = (1-p)a_0 : \text{a resonance, then we stop}$$

τ^{-1} :

$$\begin{pmatrix} 2\beta a_0 + \beta b_0 & \beta a_0 \\ 2(1-p)\beta a_0 + (1-p)\beta b_0 & (1-p)\beta a_0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} (\beta + \mu)a_0 \\ (1-p)\beta a_0 - (k + \mu)b_0 \end{pmatrix}$$

we have $a_1 = b_1 = 0$: not a resonance, then we continue

and we have a resonance if

$$\beta = \frac{(k + \mu)(p-1)}{p(p-2)} + \frac{k + \mu}{p}. \quad (17)$$

τ^0 :

$$\begin{pmatrix} 1 - 2\beta a_0 - \beta b_0 & -\beta a_0 \\ 2(1-p)\beta a_0 + (1-p)\beta b_0 & 1 + (1-p)\beta a_0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} -(\beta + \mu)a_1 \\ (1-p)\beta a_1 - (k + \mu)b_1 \end{pmatrix}$$

recall $a_1 = b_1 = 0$, we have a resonance if

$$p^2 - 2p + 1 = 0 \quad (18)$$

There are two identical real root of (18). Equations (8) and (9) possesses the Painlevé property for $p = 1$.

2.3 Singularity analysis of the second-order ordinary differential equation

From (8) we have

$$E = \frac{\dot{S} - \mu}{\beta S} - S + \frac{(\mu + \beta)}{\beta}. \quad (19)$$

The derivative of (19) gives

$$\dot{E} = \frac{\ddot{S}}{\beta S} - \frac{\dot{S}^2}{\beta S^2} + \frac{\mu \dot{S}}{\beta S^2} - \dot{S}. \quad (20)$$

The substitution of (19) and (20) into (9) gives the following second-order equation for $S(t)$

$$\begin{aligned} S\ddot{S} - \dot{S}^2 - p\beta\dot{S}S^2 + (k + \mu)\dot{S}S + \mu\dot{S} - \beta(k + p\mu)S^3 - \mu(k + \mu)S \\ - [\beta\mu(1-p) - (k + \mu)(\mu + \beta)]S^2 = 0 \end{aligned} \quad (21)$$

The first three terms in (21) are dominant, the exponent of the leading order term is -1 and the resonances are at ± 1 . To establish that there is consistency for the exponent at which the resonance occurs, we substitute

$$S = a_i \tau^{i-1} \quad (22)$$

into (21) to obtain

$$\begin{aligned} & (i-1)(i-2)a_i a_j \tau^{i+j-4} - (i-1)(j-1)a_i a_j \tau^{i+j-4} - (i-1)p\beta a_i a_j a_k \tau^{i+j+k-4} \\ & + (i-1)(k+\mu)a_i a_j \tau^{i+j-3} + (i-1)\mu a_i \tau^{i-2} - \beta(k+p\mu)a_i a_j a_k \tau^{i+j+k-3} \\ & - \mu(k+\mu)S a_i \tau^{i-1} - [\beta\mu(1-p) - (k+\mu)(\mu+\beta)]a_i a_j \tau^{i+j-2} = 0 \end{aligned}$$

The coefficients of τ^{-4} gives

$$2a_0^2 - a_0^2 + p\beta a_0^3 \implies a_0 = \frac{1}{p\beta} \quad (23)$$

and the coefficients of τ^{-3} gives

$$2a_0 a_1 - p\beta a_0(2a_0 a_1) - (k+\mu)a_0^2 - \beta(k+p\mu)a_0^3 = 0. \quad (24)$$

From the result in (23) the coefficient of a_1 is zero as is to be expected as this is where the resonance occurs. The terms remaining in (24) give the condition

$$k = \frac{-2p\mu}{p+1} \quad (25)$$

Subject to the constraint (25), the second order ordinary differential equation (21) has an analytic solution for $S(t)$. It follows from (19) that $E(t)$ is also analytic and from (7) that $I(t)$ is also analytic.

For $p = 1$ the condition (25) becomes

$$k + \mu = 0 \quad (26)$$

Condition (26) means that if $p = 1$ then the sum of the transition rate from exposed class to infectious class and the natural death rate is zero.

3 Lie Analysis

An n th order ordinary differential equation

$$N(x, y, y', \dots, y^{(n)}) = 0 \quad (27)$$

admits the one parameter Lie group of transformations

$$\bar{x} = x + \varepsilon \xi \quad (28)$$

$$\bar{y} = y + \varepsilon \eta \quad (29)$$

with infinitesimal generator

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \quad (30)$$

if

$$G^{[n]} N|_{N=0} = 0, \quad (31)$$

where $G^{[n]}$ is the n th extension of G given by

$$G^{[n]} = G + \sum_{i=1}^n \left\{ \eta^{(i)} - \sum_{j=0}^{i-1} \binom{i}{j} y^{(j+1)} \xi^{(i-j)} \right\} \frac{\partial}{\partial y^{(i)}}. \quad (32)$$

We say that the equation possesses the symmetry (group generator)

$$G = \xi \partial x + \eta \partial y \quad (33)$$

iff (31) holds.

The autonomous system (1) possesses the obvious Lie point symmetry ∂_t . The integrability from the point of view of Lie requires the knowledge of a three-dimensional solvable algebra. We acknowledge that system (1) is a first-order ordinary differential equations and possesses an infinite number of Lie point symmetries. Since infinity is not a satisfactory number, we apply Lie group analysis to the second order ordinary differential equation (21) and obtain non-trivial Lie point symmetries in two special cases:

case (1)

$$\beta + \mu = 0, k \neq 0$$

In this case we obtain an eight-dimensional Lie symmetry algebra, namely

$$\begin{aligned} G_1 &= -\frac{\exp[-kt]}{k} S \partial_S \\ G_2 &= S \partial_S \\ G_3 &= \frac{\exp[-kt]}{k^2} \partial_t - \frac{\exp[-kt]}{k} S \log[S] \partial_S \\ G_4 &= S \exp[S] \partial_S \\ G_5 &= \frac{\exp[kt]}{k} \log S \partial_t \\ G_6 &= \log[S] \partial_t - k S \log[S]^2 \partial_S \\ G_7 &= \frac{\exp[kt]}{k} \partial_t \\ G_8 &= \partial_t \end{aligned}$$

The above Lie symmetry algebra is isomorphic to $Sl(3, \mathbb{R})$. This means that equation (21) is linearizable by means of a point transformation.

case (2)

$$\beta \neq k + \mu, p \neq 1$$

This case provides the coefficients of the infinitesimal generator below:

$$\begin{aligned} \xi(t, S) &= C_1 + \exp[(\beta - k - \mu + p)t] C_2 + \frac{\exp[t] C_3}{S} + \exp[(\beta - k - \mu + p)t] S^2 C_4 \\ &\quad + \exp[t] S^2 C_5 + S C_6 - \exp[(\beta - k - \mu + p)t] (C_7 - C_8) \end{aligned} \quad (34)$$

$$\begin{aligned} \eta(t, S) &= -\exp[(\beta - k - \mu + p)t] S C_2 - \frac{(\beta - k - \mu + p) \exp[-\mu t] C_3}{S} \\ &\quad + \exp[(\beta \mu - p)t] S [(\beta \mu - p) C_7 - \mu C_8] \end{aligned} \quad (35)$$

which mean that, if $\beta \neq k + \mu, p \neq 1$, the eight-dimensional Lie symmetry algebra is generated

by the following eight operators:

$$\begin{aligned}
G_1 &= \partial_t, \\
G_2 &= \exp [(\beta - k - \mu)t] \partial_t - \exp [(\beta - k - \mu + p)t] S \partial_S, \\
G_3 &= \frac{1}{S} \exp [t] \partial_t - \frac{(\beta - k - \mu + p) \exp [-\mu t]}{S} \partial_S, \\
G_4 &= \exp [\beta - k - \mu t + p] S^2 \partial_t, \\
G_5 &= \exp [t] S^2 \partial_t, \\
G_6 &= S \partial_t, \\
G_7 &= -\exp [(\beta - k - \mu + p)t] \partial_t + (\beta \mu - p) \exp [(\beta \mu - p)t] S \partial_S, \\
G_8 &= \exp [(\beta - k - \mu + p)t] \partial_t - \mu S \exp [\beta \mu - p] t \partial_S.
\end{aligned} \tag{36}$$

In our analysis we assume that $p = 1$. For the case $p \neq 1$ the result is given by the numerical simulations [3]. However, if $\beta = k + \mu$ (see equation (17) if $p = 1$), then the eight Lie symmetries are

$$\begin{aligned}
G_1 &= \partial_t, \\
G_2 &= \exp [t] [\partial_t - S \partial_S], \\
G_3 &= \frac{1}{S} \exp [t] \partial_t - \frac{\exp [-\mu t]}{S} \partial_S, \\
G_4 &= S^2 \partial_t, \\
G_5 &= \exp [t] S^2 \partial_t, \\
G_6 &= S \partial_t, \\
G_7 &= -\partial_t + [(k + \mu)\mu - p] \exp [(k + \mu)\mu - 1] t] S \partial_S, \\
G_8 &= \partial_t - \mu S \exp [k + \mu - 1] t \partial_S.
\end{aligned} \tag{37}$$

We note that $[G_1, G_2]_{LB} = G_2$ gives a reduction of (21) by

$$G_2 = e^t (\partial_t - S(t)) \partial_S. \tag{38}$$

The associated Lagrange's system for the zeroth and first-order invariant of G_2 in (21) is

$$\frac{dt}{1} = \frac{dS}{-S} = \frac{d\dot{S}}{-2\dot{S} - S}$$

so that

$$T = t + \log S, \quad U = \frac{\dot{S}}{S^2} + \frac{1}{S} \tag{39}$$

with U and T the new dependent and independent variables, respectively. Therefore equation (21) becomes

$$\frac{dU}{dT} + U + 1 = 0$$

which can be easily integrated to give

$$(U + 1) \exp [T] = A. \tag{40}$$

Equation (40) becomes

$$(U + 1) = A \exp [-T]. \tag{41}$$

The substitution of (39) into (41) gives

$$\frac{\dot{S}}{S} + S + 1 = A \exp[-t]. \quad (42)$$

We integrate (42) to obtain

$$S(t) = \frac{AB \exp[-t]}{D \exp[A \exp[-t]] + B}. \quad (43)$$

The derivative of (42) gives

$$\begin{aligned} \dot{S}(t) &= -\frac{AB \exp[-t]}{[D \exp[A \exp[-t]] + B]} \\ &\quad - \frac{A^2 B D \exp[-2t] \exp[A \exp[-t]]}{[D \exp[A \exp[-t]] + B]^2}. \end{aligned} \quad (44)$$

Substituting (43) and (44) into (19) we have

$$\begin{aligned} E(t) &= \frac{(\beta + \mu)}{\beta} - \frac{\mu[D \exp[A \exp[-t]] + B]}{AB \beta \exp[-t]} \\ &\quad - \frac{A \exp[-t][D \exp[A \exp[-t]] + B] + B}{\beta[D \exp[A \exp[-t]] + B]}. \end{aligned} \quad (45)$$

The substitution of (43) and (45) into (7) gives

$$\begin{aligned} I(t) &= 1 - \frac{AB \exp[-t]}{D \beta \exp[A \exp[-t]] + B \beta} + \frac{\mu[D \exp[A \exp[-t]] + B]}{AB \exp[-t]} \\ &\quad + \frac{A \exp[-t][D \exp[A \exp[-t]] + B] + B}{\beta[D \exp[A \exp[-t]] + B]} \\ &\quad + \frac{AB + (\beta + \mu)}{\beta}. \end{aligned}$$

The case $k + \mu = 0$ (see equation (26) if $p = 1$) reduces equation (21) to

$$S\ddot{S} - \dot{S}^2 - \beta \dot{S}S^2 + \mu \dot{S} = 0 \quad (46)$$

The only condition for equation (46) to pass the Painlevé test is when $\mu = 0$ [15] and from (26) we have $k = 0$ as it did for the system of three first order differential equations (1). A standard method for treating a nonlinear equation such as (46) is to raise it to a higher order by means of the following Ricatti transformation [15]

$$S = -\frac{1}{\beta} \frac{\dot{\omega}}{\omega} \quad (47)$$

When the transformation (47) is applied to (46), we obtain

$$\frac{1}{\beta^2} \left(\frac{\ddot{\omega}\dot{\omega} - \dot{\omega}^2 + \mu\ddot{\omega}\dot{\omega}}{\omega^2} \right) - \frac{\mu}{\beta^2} \frac{\dot{\omega}^3}{\omega^3} = 0 \quad (48)$$

with $\mu = 0$ we obtain a generalised Kummer-Schwaz equation [15]

$$\ddot{\omega}\dot{\omega} - \dot{\omega}^2 = 0 \quad (49)$$

Equation (49) has three Lie point symmetries [15], namely

$$\begin{aligned} G_1 &= \partial_t, \\ G_2 &= \partial_\omega \\ G_3 &= \omega \partial_\omega \end{aligned} \quad (50)$$

The third symmetry G_3 is unexpected [11] and arises since the symmetry associated with the Ricatti transformation is not the normal subgroup of the two symmetries ∂_ω and $\omega\partial_\omega$. One reduces (49) to (46) using $\omega\partial_\omega$, ∂_ω which ceases to be a point symmetry [15]. Rather it becomes the exponential nonlocal symmetry [10] $S \exp[\beta S dt] \partial_S$ and ∂_ω is a Type I hidden symmetry.[2]

We reduce equation (49) to a second-order ordinary differential equation by choosing the operator G_2 . The variables for the reduction are

$$T = t, \quad W = \log \dot{\omega} \quad (51)$$

we obtain the linear second order order differential equation

$$\ddot{W} = 0 \quad (52)$$

Equation (52) has eight Lie point symmetries with the algebra $sl(3, R)$. The solution of (52) is

$$W(T) = AT + B \quad (53)$$

using the change of variables (51) we obtain

$$\omega(t) = B \int \exp[At] dt + C \quad (54)$$

therefore from (47) we have

$$S(t) = -\frac{1}{\beta} \frac{\exp[At]}{\int \exp[At] dt + B} \quad (55)$$

Hence, from (7) and (19) we have

$$\begin{aligned} E(t) = & \frac{\beta + \mu}{\beta} - \frac{-A(1 + Be^{-t}) \exp[-t + Be^{-t}]}{A\beta \exp[-t + Be^{-t}]} - \frac{A\mu \exp[-t + Be^{-t}]}{\beta(A \int \exp[-t + Be^{-t}] dt + C)} \\ & - \frac{-A(\exp[-t + Be^{-t}])^2}{\beta \exp[-t + Be^{-t}] \times (A \int \exp[-t + Be^{-t}] dt + C)} - \frac{A \exp[-t + Be^{-t}]}{(A \int \exp[-t + Be^{-t}] dt + C)} \end{aligned}$$

and

$$\begin{aligned} I(t) = & 1 - \frac{A \exp[-t + Be^{-t}]}{A \int \exp[-t + Be^{-t}] dt + C} - \frac{\beta + \mu}{\beta} + \frac{-A(1 + Be^{-t}) \exp[-t + Be^{-t}]}{A\beta \exp[-t + Be^{-t}]} \\ & + \frac{A\mu \exp[-t + Be^{-t}]}{\beta(A \int \exp[-t + Be^{-t}] dt + C)} + \frac{A \exp[-t + Be^{-t}]}{(A \int \exp[-t + Be^{-t}] dt + C)} \\ & + \frac{-A(\exp[-t + Be^{-t}])^2}{\beta \exp[-t + Be^{-t}] \times (A \int \exp[-t + Be^{-t}] dt + C)} \end{aligned}$$

4 Discussion

In this Section we study in detail the solutions in closed form that we have obtained in the cases $\beta = k + \mu$ and $k + \mu = 0$. We plot our solutions with the help of the graphing capability of MATHEMATICA. Note that the qualitative description by Blower *et al* is limited to the study of equilibrium points, their stability and bifurcation diagrams. We use the same numerical values of k and μ as given in [3]. The numerical value of β is derived from the relationship that Lie group analysis has determined.

In all the figures, the red line represents the plot of S , the blue line represents the plot of E and the green line represents the plot of I .

In the case where the infection rate is the sum of the death rate and the rate of progression from latent TB to active TB. We simulate the dynamics of the model by assuming three different values of k within the range considered in [3]. In Figure 1 and Figure 2 we show the dynamics of the model if $\beta = 0.50357$ and $\beta = 0.50527$ respectively.

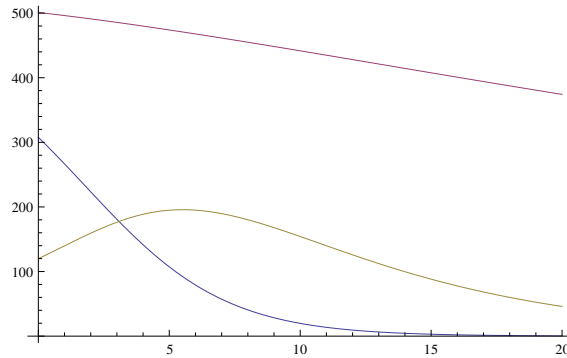


Figure 1. $\beta = 0.50357$, $k = 0.00357$, $\mu = 0.5$.

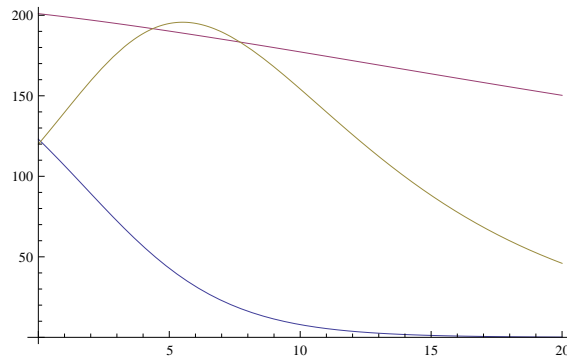


Figure 2. $\beta = 0.50527$, $k = 0.00527$, $\mu = 0.5$.

5 Conclusion

We applied the method of Lie Symmetry and Singularity analysis to a Mathematical Model which describes Tuberculosis infection. Lie group analysis is indeed the most powerful tool to find the general solution of ordinary differential equations. The Singularity analysis of the model reveals that at the resonance $r = \pm 1$, the parameter p must be one for the system to pass the Painlevé test. Lie symmetry analysis allowed us to integrate the TB model by quadrature and we found the general solution of the model if the infections rate of TB patients is the sum of the death rate of TB infecteds plus the rate of progression from latent to active TB.

References

- [1] Ablowitz M., Ramani A., Segur H., A connection between nonlinear evolution equations and ordinary differential equations of P-type II *Journal of Mathematical Physics* **21** (1980), 1006–1015.

- [2] Abraham-Shrauner B., Govinder K.S., Leach P.G.L., Integration of second order equations not possessing point symmetries *Physics Letters A* **203** (1995), 169–174.
- [3] Blower S.M., McLean A.R., Porco T.C., Small P.M., Hopwell P.C., Sanchez M.A., Moss A.R., The intrinsic transmission dynamics of tuberculosis epidemics *Nature Medicine* **1**(8) (1995), 815–821.
- [4] Blower S.M., Small P.M., Hopwell P.C., Control strategies for tuberculosis epidemics: New models for old problems *Science* **273** (1996), 497–500.
- [5] Blower S.M., Gerberding J.L., Understanding, predicting and controlling the emergence of drug-resistant tuberculosis: A theoretical framework *Journal of Molecular Medicine* **76** (1998), 624–636.
- [6] Chazy J., Sur les equations différentielles du troisième ordre et d'ordre supérieure dont l'intégrale générale a ses points critiques fixés, *Acta Mathematica* **34** (1911), 317–385.
- [7] Conte R., The Painlevé Property: One Century Later Eds. Springer-Verlag, New York, 1999.
- [8] Gambier B., Sur les equations différentielles du second ordre et premier degré dont l'intégrale générale est a ses points critiques fixés *Acta Mathematica* **33** (1889), 1–55.
- [9] Garnier R., Sur les equations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieure dont l'intégrale générale a ses points critiques fixes *Annales Scientifiques de École Normale Supérieure* **XXIX** (1912), 1–126.
- [10] Géronomi C., Contribution à l'Intégrabilité des Équations Différentielles Ordinaires Possédant des Symmetries d'Invariance par Translation et de Redimensionnement (thèse, Université d'Orléans, France), 1998.
- [11] Govinder K.S., Leach P.G.L., A group theoretic approach to a class of second order ordinary differential equations not possessing Lie point symmetries *Journal Physics A* **30** (1997), 2055–2068.
- [12] Govinder K.S., Leach P.G.L., A group theoretic approach to a class of second order ordinary differential equations not possessing Lie point symmetries *Journal Physics A* **30** (1997), 2055–2068.
- [13] Hua D.D., Cairó L., Feix M.R., Govinder K.S., Leach P.G.L., Connection between the existence of first integrals and the Painlevé property in two-dimensional Lotka-Volterra and Quadratic Systems *Proceedings of the Royal Society* **452** (1996), 859–880.
- [14] z Kowalevski S., Sur le probleme de la rotation d'un corps solide autour d'un point fixé, *Acta Mathematica* **12** (1880), 177–232.
- [15] Leach P.G.L., Lemmer R.L., The Painlevé test, hidden symmetries and the equation $y + yy' + ky^3 = 0$ *Journal of Physics A: Mathematical General* **26** (1993), 5017–2054.
- [16] Mohammad A.A., Can M., Painleé Analysis and Symmetries of the Hirota-Satsuma Equation *Journal of Nonlinear Mathematical Physics* **3** (1996), 152–155.
- [17] Nucci M.C., Leach P.G.L., Singularity and Symmetry analyses of mathematical models of epidemics *South African Journal of Science* **105** (2009), 136–146.
- [18] Ramani A., Grammaticos B., Bountis T., The Painlevé property and singularity analysis of integrable and nonintegrable systems *Physics Reports* **180** (1991), 159–245.
- [19] Song B., Castillo-Chavez C., Aparicio J.P., Tuberculosis models with fast and slow dynamics. The role of close and casual contacts *Mathematical Biosciences* **180** (2002), 187–205
- [20] Tabor M., *Chaos and Integrability in Nonlinear Dynamics* John Wiley and Sons, New York, 1989.